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SOME APPLICATIONS OF THE METHOD OF ABRIDGED NOTATION.

By Maxime Bôcher.

1. Introduction. In proving a geometrical proposition by the method of abridged notation certain identities, often of an extremely simple character, are used, from whose geometric interpretation the proof of the theorem follows. It frequently happens that the set of identities we have used in the proof of our theorem admits of a second geometrical interpretation, and when we have noticed this fact, we have before us a new geometrical theorem which we have, so to speak, deduced from the theorem from which we started, or rather from our method of proving it.

To illustrate what I mean by a very simple example, let us consider the theorem that the bisectors of the angles of a triangle meet in a point. To prove this by the method of abridged notation we indicate the equations of the sides of the triangle by

$$(1) \qquad u = 0, \qquad v = 0, \qquad w = 0.$$

If we suppose that all these equations are written in the normal form:

$$x \cos \alpha + y \sin \alpha - p = 0,$$

the equations of the bisectors of the angles will be

$$(2) \qquad u - v = 0, \qquad v - w = 0, \qquad w - u = 0;$$

and now the identity:

$$(3) \qquad (u - v) + (v - w) + (w - u) \equiv 0$$

shows us at once that these lines meet in a point.

Suppose, however, instead of interpreting the equations (1) as the equations of three straight lines, we regard them as the equations of three circles all written in the normal form:

$$x^2 + y^2 + ax + by + c = 0.$$

Then the equations (2) will represent the common chords of these circles, and the identity (3) tells us that these common chords meet in a point.

Thus we see that the method of abridged notation may be used to deduce new theorems from known ones, just as the methods of projection, reciprocation, and inversion are commonly used. As this fact has, in spite of its elementary nature, gained as yet no place, so far as I have found, in even the best text books,* I propose to illustrate it here by first reproducing the well known proofs by the method of abridged notation of the harmonic properties of the complete quadrilateral and of Desargues's theorem, and then interpreting these proofs in such a way as to obtain theorems which, instead of referring to rectilinear figures, as did the original propositions, refer to much more general curvilinear figures. In doing this I have paid special attention to the case in which these curved lines are circles, not only because this is the simplest case, but because the theorems thus obtained are suited to hold a fundamental position, as I expect to show on another occasion, in a systematic development of what has been termed the Geometry of Inversion.† It must, however, be clearly understood that the generalizing process which we use in this simple case is not identical with the process of inversion, but that the results obtained include as a very special case the theorems which the method of inversion would give us.

2. Rectilinear Figures. Let us denote the sides of a complete quadrilateral by the equations:

$$(4) \qquad \alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad \delta = 0.$$

Denoting the six vertices by

$$(5) \qquad (\alpha, \beta), \qquad (\gamma, \delta), \qquad (\alpha, \gamma), \qquad (\beta, \delta), \qquad (\alpha, \delta), \qquad (\beta, \gamma),$$

we will let

(6)
$u = 0$ represent the diagonal joining (β, γ) and (α, δ),
$v = 0$ " " " " (α, γ) " (β, δ),
$w = 0$ " " " " (α, β) " (γ, δ).

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* Except in the case where we merely pass from point to line coordinates,—a case which is essentially equivalent to reciprocation.

† *Cf.* Klein's *Erlanger Programm* (1872), reprinted in English translation and with a few additions in the *Bulletin of the New York Mathematical Society* for July 1893. A German reprint with still further additions will be found in the *Math. Annalen*, vol. 43 (1893), p. 63.

The values of u, v, w, in terms of α, β, γ, δ may be obtained as follows :— We may write the equation of the diagonal u in either of the forms

$$b\beta + c\gamma = 0, \qquad - a\alpha - d\delta = 0.$$

If we so choose the constants a, b, c, d that the first members of these equations are equal, not merely proportional, we get the identity :

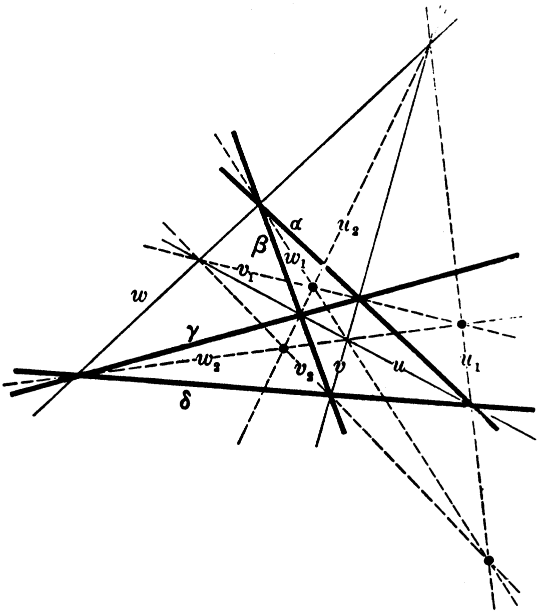


FIG. 1.

$$(7_1) \qquad u \equiv b\beta + c\gamma \equiv - a\alpha - d\delta.$$

From this it follows that

$$b\beta + d\delta \equiv - a\alpha - c\gamma.$$

If we set this expression equal to zero we get, as we see from the left hand side, a line through (β, δ), as we see from the right hand side, a line through (α, γ). We may therefore write :

$$(7_2) \qquad v \equiv b\beta + d\delta \equiv - a\alpha - c\gamma.$$

Similarly we get :

$$(7_3) \qquad w \equiv c\gamma + d\delta \equiv - a\alpha - b\beta.$$

Finally indicating the lines which join the vertices of the quadrilateral with the points of intersection of the diagonals, as shown in the figure, by:

$$(8)\qquad u_1 = 0, \qquad u_2 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \qquad w_1 = 0, \qquad w_2 = 0,$$

we have

$$(9)\qquad \begin{aligned} u_1 &\equiv v + w \equiv -a\alpha + d\delta, \\ u_2 &\equiv v - w \equiv \quad b\beta - c\gamma, \\ v_1 &\equiv w + u \equiv -a\alpha + c\gamma, \\ v_2 &\equiv w - u \equiv \quad d\delta - b\beta, \\ w_1 &\equiv u + v \equiv -a\alpha + b\beta, \\ w_2 &\equiv u - v \equiv \quad c\gamma - d\delta. \end{aligned}$$

From these formulæ we read off at once the harmonic properties of the complete quadrilateral, for instance that the two sides α and β are separated harmonically by the diagonal w through their point of intersection and the line w_1 which joins their point of intersection with the intersection of the two other diagonals (*cf.* (7_3) and the fifth formula (9)).

We may add that formulæ (9) also show that the six lines (8) pass three by three through four points, thus forming the sides of a complete quadrangle.

We pass now to Desargues's theorem:—

If two triangles are so situated that the points of intersection of corresponding sides are collinear, then the lines joining corresponding vertices are concurrent.

To prove this theorem we denote the sides of the first triangle by:

$$(10)\qquad u_1 = 0, \qquad u_2 = 0, \qquad u_3 = 0;$$

the sides of the second by:

$$(11)\qquad v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 0.$$

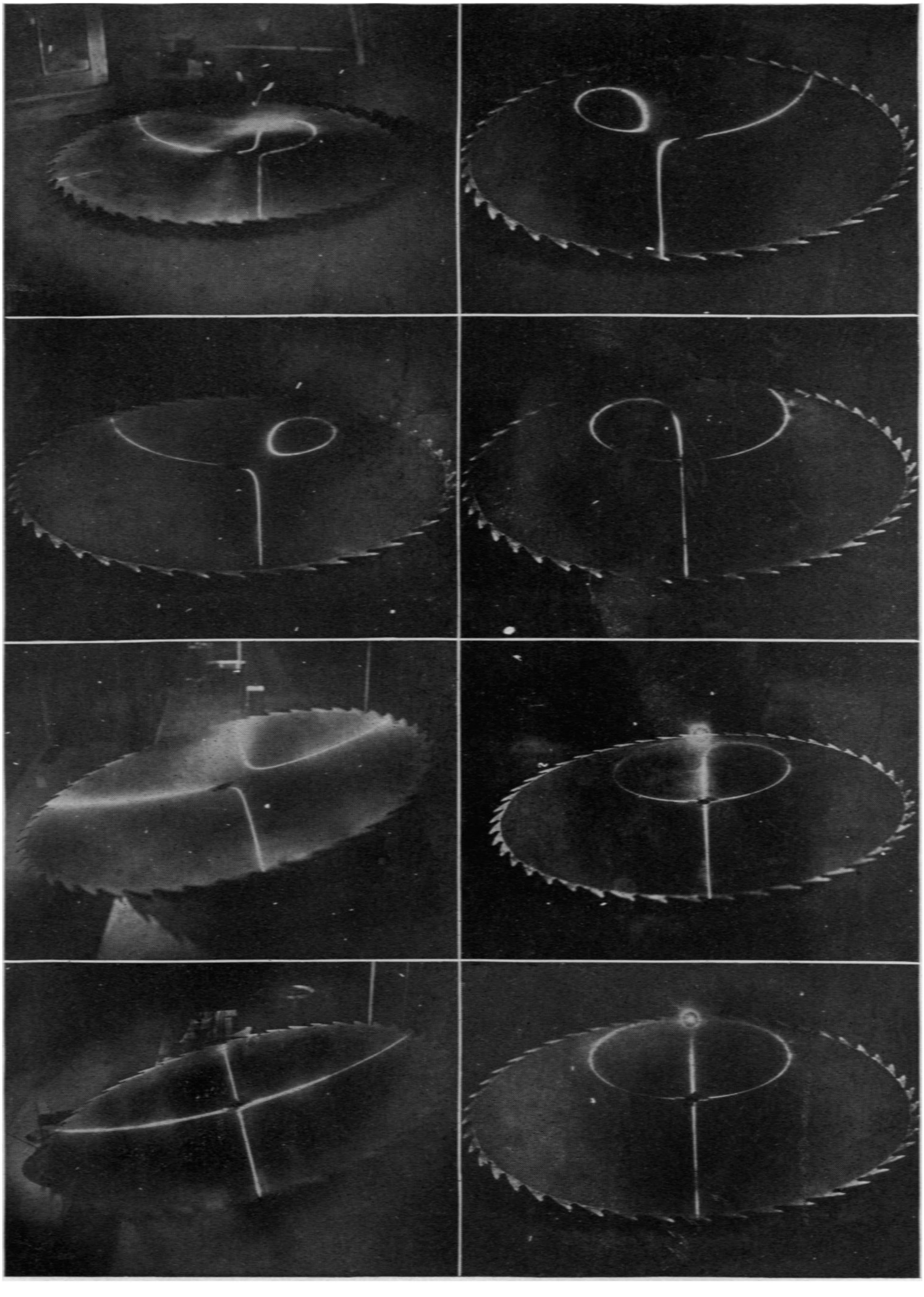
The line on which the points of intersection of corresponding sides lie may be represented by an equation in any one of the following forms:

$$a_1u_1 + b_1v_1 = 0, \qquad a_2u_2 + b_2v_2 = 0, \qquad a_3u_3 + b_3v_3 = 0.$$

Accordingly, by properly choosing the constants, we can make the first members of these equations identically equal:

$$(12)\qquad a_1u_1 + b_1v_1 \equiv a_2u_2 + b_2v_2 \equiv a_3u_3 + b_3v_3.$$

From this we get at once the following three identities:



$$
(13)\qquad
\begin{aligned}
w_3 &\equiv a_1u_1 - a_2u_2 \equiv -b_1v_1 + b_2v_2,\\
w_1 &\equiv a_2u_2 - a_3u_3 \equiv -b_2v_2 + b_3v_3,\\
w_2 &\equiv a_3u_3 - a_1u_1 \equiv -b_3v_3 + b_1v_1.
\end{aligned}
$$

From these identities we infer that the equations:

$$(14)\qquad w_1 = 0, \qquad w_2 = 0, \qquad w_3 = 0$$

represent the lines joining corresponding vertices of the two triangles, and since

$$(15)\qquad w_1 + w_2 + w_3 \equiv 0,$$

these lines are concurrent.

3. Generalization to Circles. Let us now try to interpret the analytical work of the preceding section on the supposition that the equations there used represent not straight lines, but circles.

We begin with the theorems about the complete quadrilateral.

The figure formed by four circles, no three of which meet in the same two points, we will call a complete circular quadrilateral, the circles (4) being the *sides* of the quadrilateral and the six pairs of points (5) being its six *sets of vertices*. These latter arrange themselves in pairs of *opposite* sets of vertices. The equations $b\beta + c\gamma = 0$ and $-a\alpha - d\delta = 0$ now represent circles through the pairs of points (β, γ) and (α, δ) respectively, but it will not be possible in general to make these two circles coincide, since a circle cannot in general be passed through two pairs of points.

In order that our analytic work be applicable to this case it is therefore necessary that the four points (β, γ) and (α, δ) be concyclic (*i. e.* lie on a circle). This circle will then be represented by the equation $u = 0$ (*cf.* (7_1)). From this equation we infer as before that $v = 0$ and $w = 0$ (*cf.* (7_2) and (7_3)) represent circles through the two pairs of points (α, γ), (β, δ) and (α, β), (γ, δ) respectively. We thus get the theorem:

*If in a complete circular quadrilateral two opposite sets of vertices are concyclic, the same will be true of each of the other pairs of opposite sets of vertices.**

If a circle passes through a pair of opposite sets of vertices of a complete

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* Another form of statement for this theorem which brings it into relation with Desargues's theorem is the following, in which we denote the figure formed by two circles as a *crescent*, the two circles being the *sides* and their points of intersection the *vertices*:

If two crescents are so situated that their vertices are concyclic, the points of intersection of circular quadrilateral, we will call it a diagonal of this quadrilateral. We may then state the last theorem as follows:

If a complete circular quadrilateral has one diagonal, it has three.

We will recall here two terms which we shall find it convenient to use. $S_1 = 0$ and $S_2 = 0$ being any two given circles, we call the simply infinite family

$$k_1 S_1 + k_2 S_2 = 0 \tag{16}$$

a *pencil* of circles; and we use the same term for a finite number of circles of this family. Similarly $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ being any three circles which do not belong to a pencil the doubly infinite family:

$$k_1 S_1 + k_2 S_2 + k_3 S_3 = 0 \tag{17}$$

we call a *net*.*

We now infer at once from formulæ (7):

A necessary and sufficient condition that a complete circular quadrilateral have diagonals is that its sides belong to a net. If this condition is fulfilled, the diagonals belong to the same net.

Under these conditions any two sides of the quadrilateral (as α, β) and the two diagonals which do not pass through the intersections of these sides (as u, v) form a complete circular quadrilateral whose sides belong to a net and which therefore has diagonals. Two of these diagonals are the other two sides (γ, δ) of the original quadrilateral, but the third is the circle (w_1) through the points of intersection of α with β and of u with v. Similarly we obtain the other circles (8).

By the *cross ratio* of a pencil of four circles we shall understand the cross ratio of the pencil of tangents at any one of their points of intersection.† With

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corresponding sides (the correspondence being taken in either one of the two possible ways) are also concyclic.

Still another form, in which, however, only half the theorem is explicitly stated, is the following:

Through each of the sets of vertices of a circular triangle a circle is passed. If these three circles meet in a first point, they will meet again in a second point.

As thus stated the theorem is seen to be identical with the generalization by inversion of the theorem that the common chords of three circles meet in a point.

* We mention in passing that such a net consists of all circles which cut at right angles the common orthogonal circle of S_1, S_2, S_3. *Cf.* for instance Casey's *Analytical Geometry*, p. 107.

† Or more generally the cross ratio of the pencil formed by the polars of an arbitrarily chosen point. This more general definition is necessary if the circles of the pencil touch each other.

this definition it is readily seen that a necessary and sufficient condition that the circles S_1 and S_2 be divided harmonically by $S_1 + \lambda S_2$ and $S_1 + \mu S_2$ is that $\lambda = - \mu$. We thus obtain, precisely as in §1, the harmonic properties of the complete circular quadrilateral, of which the following is typical:

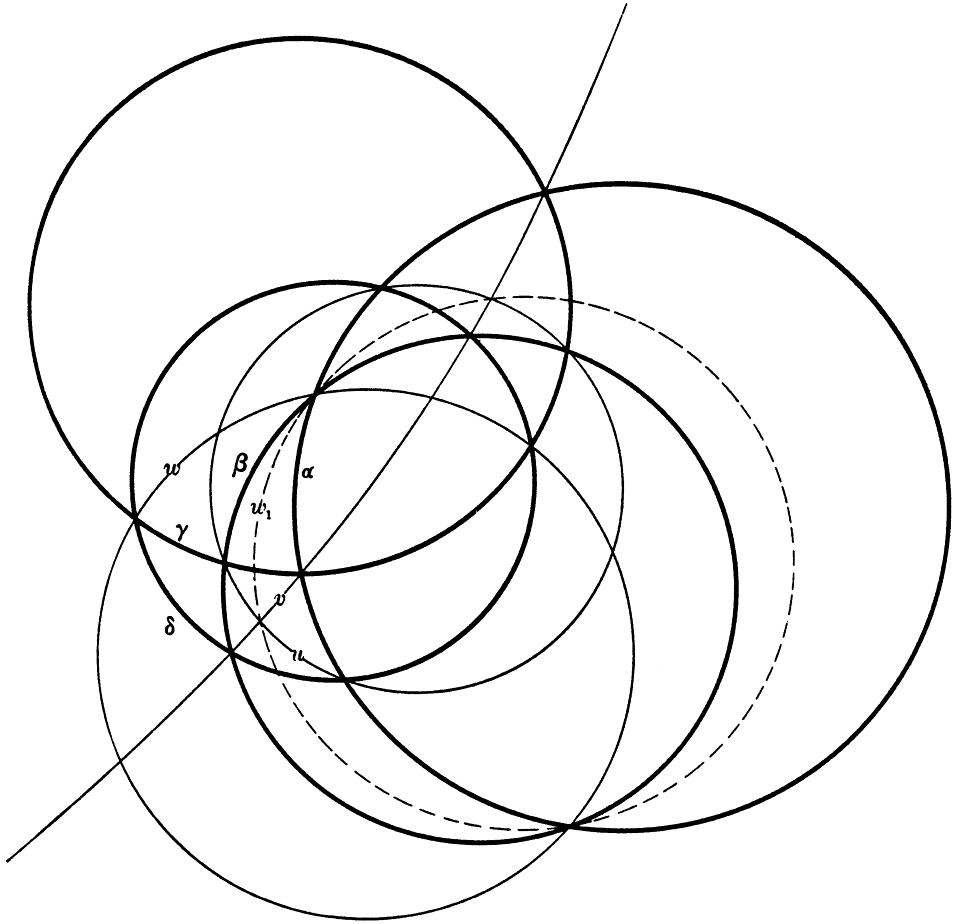

FIG. 2.

In a complete circular quadrilateral whose sides belong to a net, any two sides are divided harmonically by the diagonal through their points of intersection, and the circle through their points of intersection and through the points of intersection of the other two diagonals.

We leave to the reader the extension of the theorem that the lines (8) form a complete quadrangle.

Up to this point all the circles we have used have belonged to a single net;* and it might seem at first sight as though the method we are using must necessarily always give us theorems which involve only circles of a single net, since the straight lines from which these circles were obtained themselves form a net. This, however, is not the case, as the extension of Desargues's theorem will now show. For although any four lines of the plane are connected by a relation of the form $a\alpha + b\beta + c\gamma + d\delta \equiv 0$, it is not at all necessary that this relation be involved in the formulæ we use in proving our proposition, as is shown by the proof we have given of Desargues's theorem.†

Turning now to the proof of Desargues's theorem given in §1 we have to consider two circular triangles (10) and (11). We understand of course by a circular triangle the figure formed by three circles which do not form a pencil. Such a triangle has three sides and three pairs of vertices. The analysis of §1 then gives us the theorem:

If two circular triangles are so situated that the three pairs of points of intersection of corresponding sides are concyclic, then any pair of vertices of the first triangle and the corresponding pair of vertices of the second are concyclic,‡ and the three circles thus determined form a pencil.

The ten circles involved in this theorem do not necessarily all belong to a net, since the three sides of the first triangle and one side of the second may be arbitrarily chosen.

4. Curves of the n-th Order. We will now regard the equations from which we started as representing plane curves of the nth order instead of straight lines as in §2 or circles as in §3. We will speak of pencils and nets of curves of the nth order, using the words in the ordinary sense similar to that explained for circles in §3. We will speak of four curves of the nth order, no three of which form a pencil, as the *sides* of a *complete curvilinear quadrilateral.*[*] Each pair of sides of this quadrilateral determines a pencil, and if the pencils determined by two opposite pairs of sides have a curve in common, we will speak of this curve as a *diagonal* of the quadrilateral. Formulæ (7) now establish the following theorem:

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* In fact our formulæ practically establish a one-to-one transformation of the straight lines of the plane into the circles of a net, and therefore a one-to-two point transformation (a point determined by two straight lines into the pair of points determined by the corresponding circles). Such a correspondence can easily be established geometrically by projecting the plane from an arbitrarily chosen point onto an arbitrarily chosen spherical surface, and then projecting back stereographically onto the plane.

† A different proof of this theorem which does not have this advantage is given in Salmon's *Conic Sections*, p. 59.

‡ This first part of the theorem does not go beyond what we have already proved, since two pairs of corresponding sides of the triangle form a complete circular quadrilateral lying wholly in a net.

A necessary and sufficient condition that a complete curvilinear quadrilateral have a diagonal is that its sides belong to a net, and if this condition is fulfilled, it has three and only three diagonals which also belong to the net.

Let us now consider the pencil of four curves

$$S_1 = 0, \qquad S_2 = 0, \qquad S_1 + \lambda S_2 = 0, \qquad S_1 + \mu S_2 = 0.$$

The linear polars of any point P of the plane, excluding possibly certain exceptional positions for P, are easily shown to form a pencil, and the cross ratio of this pencil is seen to be independent of the position of P and to have the value λ/μ. This quantity we will speak of as the cross ratio of the pencil of curves.[†] In particular we may in general[‡] take for P a point of intersection of the curves, in which case the cross ratio of the pencil of curves is given by the cross ratio of the pencil of tangents at this point of intersection.

If $\lambda = -\mu$ we say that the curves S_1 and S_2 are *divided harmonically* by the other two curves.

Formulæ (7) and (9) now give us a number of theorems concerning the complete curvilinear quadrilateral, of which we state the following one:

If a complete curvilinear quadrilateral is formed by four curves of a net, any two sides α, β of the quadrilateral are divided harmonically by the diagonal belonging to the pencil (α, β) and the curve common to the pencil (α, β) and the pencil determined by the other two diagonals.

The extension to this case of the further theorems contained in §2 we leave to the reader.

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[*] Such a quadrilateral has in general six sets of n^2 vertices each. In case some of the sides are reducible curves two sides may, however, have a whole curve in common. For this reason we have avoided speaking of vertices in the text.

[†] If instead of considering the linear polars of P we consider the polars of any order, it is clear that these will also form a pencil having the same cross ratio as that of the original pencil.

[‡] Provided, namely, there exists a point of intersection where neither S_1 nor S_2 has a double point, and where these curves do not touch each other.

5. Extension to Surfaces. It is hardly necessary to do more than mention the possibility of interpreting the analysis of §2 on the supposition that the equations there used represent surfaces in space instead of lines or curves in the plane. Thus, for instance, if we regard the equations as representing spheres, the formulæ concerning the complete quadrilateral in §2 give us certain theorems concerning spheres which have two points in common, while the proof of Desargues's theorem gives us a theorem concerning ten spheres which cut a given sphere orthogonally. The statement of these theorems and of similar theorems concerning surfaces of any order presents no difficulty.

Even without changing the degrees of the equations we use, the method here considered gives us the means of transferring theorems in plane geometry to the case of three dimensions. The theorems of §2 are too simple to give anything interesting in this way; but it will be a good exercise for a student to transfer by this method to quadric surfaces the theorem that *if three conics have double contact with a fourth, six of their chords of intersection pass three by three through four points*, using for this purpose the proof by abridged notation given in Salmon's *Conic Sections*, p. 243.

It will be seen that examples of cases to which our method applies can be multiplied indefinitely. We have confined ourselves here to a few of the very simplest illustrations.

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